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Constructive Approach to Logics of Physical Systems: Application to EPR Case

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A constructive approach to logics of physical systems, according to which families of propositions about physical systems are not defined in an axiomatic way, but are built up in the course of experiments, is proposed. Several ways of joining Boolean algebras of propositions obtained in single experiments are studied. The proposed approach is applied to study families of propositions encountered in EPR-type experiments. Two examples of such experimental families of EPR propositions are studied and they are compared with two theoretical families of EPR propositions in the literature.

"The prime source of scientific knowledge about the physical world is the experience gained by systematic observation of physical systems."
J. M. Jauch (1968)

1. INTRODUCTION

In the overwhelming majority of papers on the quantum logic approach, all mathematical structures which describe physical systems are once and for all established after adopting a set of more or less physically plausible axioms. The choice of axioms varies from one author to another, but usually it is not hidden (see, e.g., Mackey, 1963; Piron, 1976; Beltrametti and Cassinelli, 1981) that one of the aims is to obtain structures which mirror those encountered in already existing theories: an orthomodular lattice of closed subspaces of a Hilbert space in quantum mechanics or a Boolean algebra of subsets of a phase space in classical mechanics. Therefore, the choice of axioms is strongly biased by the already existing theories and a kind of a vicious circle results.

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In the present paper we continue, with slight modifications, the line of investigation originated in Posiewnik and Pykacz (1986). We analyze in quantum-logical terms the process of building a theory without any previous knowledge about the system. This is done step by step by making experiments or observations and accumulating data. For this reason, contrary to what is usually encountered in the literature, nothing like the notion of the set of all propositions appears in our approach in the very beginning. New propositions are added to the set of already existing ones if new experiments, testing new properties of a system, are performed, and this process may terminate or not. This explains the word "constructive" in the title of the paper.

2. STAGE OF COLLECTING EXPERIMENTAL DATA WITHOUT AN UNDERLYING THEORY

Our starting point will be the concept of an experiment. This consists of a set of well-defined manipulations (or observations) which allow one to give unambiguously the answer "yes" or "no" to every sensible question. The word "question" is used here in its linguistic sense. For example: "Is the pointer of the instrument between numbers N and $N+1$ on the reading scale?" or "Is the brightness of a just observed UFO greater than the brightness of Sirius?" are both sensible questions. These examples should make it clear that the title of the present section is not meant as a requirement of the absence of any theory of our experimental device. Nevertheless, such a theory may be, at this stage, irrelevant: we can measure the brightness of a UFO with the aid of a photometer and could have a perfect theory of a photometer without any theory of the UFO.

In order to obtain something more than a bare description of a unique phenomenon, we have to assume that it is possible to repeat the experiment. To "repeat" means that we do our best to be convinced that in every run of the experiment we deal with the same physical system possessing the same features. In the most general case, since we cannot exclude the situation that some experiments could alter irreversibly or even destroy the system, we admit that particular runs of the experiment can be performed on identical copies of our system. This is in clear accordance with the everyday practice of any experimentalist.

According to the spirit of the "Geneva School" (Jauch, 1968; Jauch and Piron, 1969; Piron, 1976; Aerts, 1981, 1982), a class of questions having the same answer (yes or no) in every run of the experiment will be called a *proposition.*

Let us assume that a finite number n of runs of the chosen experiment is completed. By the very definition of a proposition we can identify each proposition tested by this experiment with the n-element sequence of 0's and 1's:

$$a = (a_i), \qquad a_i = 0 \text{ or } 1 \text{ ("no" or "yes")}, \qquad i = 1, 2, \ldots, n \tag{1}$$

To avoid a misunderstanding, we stress that only rarely does a real experiment concern just one proposition. Even one measurement of a single quantity usually determines many propositions at a time. For instance, the measurement of the length of a table in millimeters determines simultaneously the truth value of the whole family of propositions of the form: "the length of a table is between N and $N+1$ millimeters." Nevertheless, since the dispersion of results in a finite sequence of experimental runs is finite, the obtained family of propositions is finite as well. The number of propositions of the form (1) obtained in n runs of the experiment cannot obviously be bigger than 2^n, since we have assumed that at this stage we do not have any theory, so the only test of whether two propositions are different consists in checking if they are represented by different sequences of 0's and 1's.

The finite set A of all propositions defined by (1) which are obtained in n runs of the experiment e is a Boolean algebra with respect to pointwise partial order

$$a \leq b \Leftrightarrow a_i \leq b_i \qquad \text{for all} \quad i = 1, 2, \ldots, n \tag{2}$$

and orthocomplementation

$$': \quad a \to a' = (1 - a_i) \tag{3}$$

Meets and joins of propositions with respect to pointwise partial order (2) can be calculated pointwisely as well:

$$\begin{aligned} a \wedge b &= (a_i \wedge b_i) = (\min(a_i, b_i)) = (a_i b_i) \\ a \vee b &= (a_i \vee b_i) = (\max(a_i, b_i)) = (a_i + b_i - a_i b_i) \end{aligned} \tag{4}$$

According to (4), we have

$$\begin{aligned} a \wedge a' &= (\min(a_i, 1 - a_i)) = (0, 0, \ldots, 0) \in A \\ a \vee a' &= (\max(a_i, 1 - a_i)) = (1, 1, \ldots, 1) \in A \end{aligned} \tag{5}$$

The elements appearing on the right-hand side of (5) represent, respectively, *absurd* and *trivial* propositions (Jauch, 1968; Piron, 1976) and they will be denoted $\varnothing$ and ι. They are, respectively, the least and the greatest elements of the Boolean algebra A.

We would like to indicate the analogy of our constructive approach with the theoretical approach in Mączyński (1973, 1974). Mączyński represented propositions by [0, 1]-valued functions defined on the set of states.

Orthogonal propositions, i.e., propositions a, b such that $a \leq b'$ [denoted $a \perp b$ in Mączyński (1973, 1974)] are represented by functions f_a and f_b such that $f_a + f_b \leq 1$. In our finite case functions are replaced by sequences according to the formula

$$a \perp b \Leftrightarrow a_i + b_i \leq 1 \qquad \text{for all} \quad i = 1, 2, \ldots, n \tag{6}$$

Mączyński proved as well that the join of a sequence of pairwise orthogonal propositions coincides with the algebraic sum of functions. This also happens in our finite Boolean algebra A, since if $\{a_\alpha\} = \{(a_{\alpha,i})\}$ is a sequence of pairwise orthogonal propositions and if $a_{\alpha,i} = 1$ for some $i = 1, 2, \ldots, n$, then, necessarily, $a_{\beta,i} = 0$ for all $\beta \neq \alpha$; therefore

$$\bigvee_\alpha a_\alpha = \left(\sum_\alpha a_{\alpha,i} \right) \qquad \text{if} \quad a_\alpha \perp a_\beta \text{ for } \alpha \neq \beta \tag{7}$$

The analogy with Mączyński's functional representation suggests that states of our system studied in n runs of the experiment e should be labeled by natural numbers $i = 1, 2, \ldots, n$. Of course, it may happen that for some $i \neq j$, $a_i = a_j$ for all $a \in A$. In such a case these two states are, at this stage of investigation, indistinguishable, or alternatively, we admit that both i and j represent the same state. Let us note that after such a procedure of identification of indistinguishable states, states correspond one to one to atoms of the Boolean algebra A, i.e., to elements which are just above $\varnothing$ in our partial order (2). This is again in full agreement with the later papers of the "Geneva School," where pure states of a physical system can be identified with atoms of the lattice of propositions. There is no contradiction between the possibility of obtaining several different pure states identified with atoms of A and the requirement that, to the best of our knowledge, the system should possess the same features in every experimental run. The case when A has several atoms means that in the course of our chosen experiment e the system was in a mixed state. Moreover, our approach provides the possibility of counting weights with which different pure states enter this mixture—a problem which is generally beyond the scope of the axiomatic quantum logic approach. Due to the following lemma, this can be simply done by counting the number of 1's in each atom.

Lemma 1. If $\{a_\alpha\}$ is the family of all atoms of the Boolean algebra A_n, then for each $i = 1, 2, \ldots, n$ there exists exactly one atom a_α such that $a_{\alpha,i} = 1$.

Proof. Every element of a finite Boolean algebra is the join of the atoms it dominates (see, e.g., Halmos, 1974). Since I dominates all elements in A, if there existed i such that $a_{\alpha,i} = 0$ for all atoms a_α, then the ith element of I should be 0 instead of 1. On the other hand, the meet of any two different atoms is the least element, so if $a_{\alpha,i} = a_{\beta,i} = 1$, then $a_\alpha = a_\beta$. Thus, the lemma follows. ■

Since each atom a_α can be identified with the proposition "the system is in the pure state a_α," the frequency of confirming this proposition in n runs of the experiment e equals the frequency of finding 1 in the sequence a_α, i.e., to the number

$$p(a_\alpha) = n^{-1} \sum a_{\alpha,i} \tag{8}$$

From Lemma 1 we infer that

$$\sum_\alpha p(a_\alpha) = n^{-1} \sum_\alpha \sum_i a_{\alpha,i} = n^{-1} n = 1 \tag{9}$$

i.e., all weights in the decomposition into pure components of the mixed state in which our system was in the course of the experiment e sum up to 1, as expected. As an example, let us assume that in the course of five runs of the experiment e, the Boolean algebra $A = \{\varnothing = (00000),\ a = (11100),\ b = (00010),\ c = (00001),\ a \vee b = (11110),\ a \vee c = (11101),\ b \vee c = (00011),\ \mathrm{I} = (11111)\}$ was obtained. Elements a, b, and c are atoms of A and we infer that the physical system in the course of the experiment e was in the mixed state $p = 0.6a + 0.2b + 0.2c$.

The same procedure of counting the fraction of 1's for each proposition $a \in A$ gives us a link with the other notion of a state often encountered in axiomatic quantum logics. According to many authors (see e.g., Mackey, 1963; Jauch, 1968; Beltrametti and Cassinelli, 1981), a state of a system is described by a *probability measure* defined on the orthocomplemented and orthocomplete partially ordered set of propositions L, i.e., by a mapping $p\colon L \to [0, 1]$ such that $p(\mathrm{I}) = 1$ and $p(\bigvee a_\alpha) = \sum p(a_\alpha)$ for any sequence of pairwise orthogonal propositions. It is easy to see that in our case the function $p\colon A \to [0, 1]$ defined by the formula (8) for any proposition $a \in A$, not only for atoms of A, is the probability measure on A. The state of a physical system described by this probability measure is of course the previously mentioned mixed state of a system.

3. JOINING TOGETHER BOOLEAN ALGEBRAS OBTAINED IN DIFFERENT EXPERIMENTS

Throughout the previous section we were dealing with the single finite Boolean algebra A obtained in n runs of the same experiment e. This was also the case studied under a slightly different aspect in Posiewnik and Pykacz (1986). Now we would like to go further and study the possibility of joining together several finite Boolean algebras obtained in different experiments performed on the same physical system. Such a situation happens, for example, when we measure (in different experiments) physical quantities which are not simultaneously measurable, such as linear polarization in different directions.

Let us assume that we have a family of finite Boolean algebras obtained by performing sequences of different experiments on identical copies of a physical system and then we join them together by identifying some elements which belong to different members of the family. The process of identification is a sign that a kind of primitive pretheory, at least in the form of a set of vague intuitions, has started to emerge in our mind, provided we had no theory of the studied phenomena before. Of course this is rarely the case nowadays. More often a theory of even some theories are already at hand and the identification of propositions is made on this basis. In both cases the natural question about the emerging structure of the set of propositions arises. The structure of the set of experimentally obtained propositions can serve as a hint for building up a theory in the first case or as a test of compatibility of already existing theories with experiments in the second case.

We shall study several kinds of such joining, starting from the simplest one. In the following we assume that all Boolean algebras are obtained by performing different experiments. The case of having more than one algebra obtained by several series of runs of the same experiment can be eliminated in the very beginning by collecting all data from all runs before we start to build the Boolean algebra of propositions. Therefore, we begin the construction having a finite family $\{A_k\}$ of finite Boolean algebras, where each subscript $k = 1, 2, \ldots, m$ represents a different experiment.

The process of joining Boolean algebras of different experiments corresponds to the introduction of some degree of noncontextuality. In an extremely contextual theory it is assumed that the measured properties may depend on the full experimental context and therefore propositions defined in different experiments are never identified. Contextual theories rest upon weakest assumptions, but they also have the least structure. The introduction of some degree of noncontextuality, therefore, can be seen as a theoretical progress.

We shall quote now, for the reader's convenience, several ways of joining together Boolean algebras studied in the mathematical literature. Most of the quoted results can be found in Kalmbach (1983).

3.1. Horizontal Sum

A horizontal sum of a family of Boolean algebras is obtained by identifying all the least and, respectively, all the greatest elements from all members of the family. This yields the structure L in which we can easily distinguish a family of maximal Boolean subalgebras, usually called *blocks* such that $A \cap B = \{\varnothing, \mathrm{I}\}$ for any pair of different blocks of L. Of course in our case each block is a Boolean algebra of propositions obtained in a single experiment. The following simple lemma can be easily proved.

Lemma 2. A horizontal sum of a family of Boolean algebras is an orthomodular lattice.

After forming a horizontal sum of experimental Boolean algebras our description of a physical system is still extremely contextual, since the only identified elements are the least and the greatest ones from every algebra, and represent nothing more than the absurd and the trivial propositions $\varnothing$ and I, which can be expressed, respectively, in the form: "the physical system does not exist" and "the physical system exists."

3.2. Joining by Identification of Atoms

Since any finite Boolean algebra is completely determined by its atoms, it is natural to study in the next step the possibilities yielded by identifying atoms from different experimental Boolean algebras. This means that we identify some pure states recognized in different experiments.

The simplest situation is encountered when two different Boolean algebras can have, besides $\varnothing$ and I, at most one atom x and its orthocomplement x' in common. In such a situation the following Loop Lemma of Greechie (1971) is a useful tool to study whether the resulting structure is an orthomodular poset, orthomodular lattice, or neither of these structures.

Lemma 3. (Loop Lemma). Let $\mathbf{B}$ be a set of Boolean algebras such that for any two different members A, B of $\mathbf{B}$ either $A \cap B = \{\varnothing, \mathrm{I}\}$ or $A \cap B = \{\varnothing, \mathrm{I}, x, x'\}$, where x is an atom in both A and B and $x'^A = x' = x'^B$. Let the partial order and orthocomplementation on the set $L = \bigcup A_i$, $A_i \in \mathbf{B}$, be induced by the elements of $\mathbf{B}$. Then (i) L is an orthomodular poset if $\mathbf{B}$ does not contain a loop of order 3, and (ii) L is an orthomodular lattice if $\mathbf{B}$ does not contain a loop of order 3 or 4, where by a loop of order n is meant a finite sequence $(B_0, B_1, \ldots, B_{n-1})$ of elements of $\mathbf{B}$ such that

$$\begin{aligned} &B_i \cap B_{i+1} \text{ consists of exactly four elements} \\ &B_i \cap B_j = \{\varnothing, \mathrm{I}\} \quad \text{for } j \neq i+1, i-1 \\ &B_0 \cap B_1 \cap B_2 = \{\varnothing, \mathrm{I}\} \quad \text{for } n = 3 \end{aligned} \tag{10}$$

(the computation of the i, j is modulo n).

We shall utilize this lemma in the next section, in which structures of sets of propositions appearing in EPR-type experiments are studied.

A more sophisticated technique is required when "neighboring" Boolean algebras can have more than one atom–coatom pair in common. We again refer the reader interested in the details of this technique, called *pasting* of the Boolean algebras, to Kalmbach (1983).

3.3. Joining by Identification of Nonatoms

In the situation in which identified elements are not necessarily atoms of Boolean algebras, the following Bundle Lemma can be of some help.

Lemma 4. (Bundle Lemma). Let **B** be a set of Boolean algebras all of which have the same $\varnothing$ and ι and such that for any two elements $A, B \in \mathbf{B}$ the set-theoretic intersection of A and B carries a subalgebra $A \cap B$ of both A and B. Let us define on $L = \bigcup A$, $A \in \mathbf{B}$, the relation $\leq$ as the union of all relations $\leq_A$, $A \in \mathbf{B}$, and let us define the orthocomplementation map by $x' = x'^A$ if x is in A. Then: (a) if $\leq$ is transitive, then $(L, \leq, ')$ is an orthocomplemented poset, (b) if $\leq$ is transitive and any two elements x, y of L have a join in L which coincides with their join in A if both belong to some A of **B**, then $(L, \leq, ')$ is an orthomodular lattice.

A similar construction was studied by Finch (1969). His definition of a logical structure, written down with the aid of notation used throughout this paper, looks as follows.

A *logical structure* is an indexed set $\mathbf{L} = \{A_\gamma, \gamma \in \Gamma\}$ of Boolean algebras with the following properties:

(i) Each A_γ has the same least element $\varnothing$.
(ii) If $x, y \in A_\alpha \cap A_\beta$, then $x \leq_\alpha y$ if and only if $x \leq_\beta y$.
(iii) If $x \leq_\alpha y$ and $y \leq_\beta z$, there is γ in Γ such that $x \leq_\gamma z$.
(iv) If x belongs to $A_\alpha \cap A_\beta$, then $x'^\alpha = x'^\beta$.
(v) If x and y belong to $A_\alpha \cap A_\beta$, then $x \vee_\alpha y = x \vee_\beta y$.
(vi) Suppose that $y \leq_\alpha x'^\alpha$ for some x and y in A_α; if $x \leq_\beta z$ and $y \leq_\gamma z$, there is A_δ which contains x, y, and z.

Finch calls the set $L = \bigcup \{A\gamma \colon \gamma \in \Gamma\}$ the *logic associated with the logical structure* **L** and proves that L, endowed with operations induced by **L**, is an orthomodular poset.

The following theorem shows that families of Boolean algebras which satisfy the assumptions of the Bundle Lemma and logical structures of Finch are closely connected.

Theorem 1. If a family **B** of Boolean algebras satisfies the assumptions of version (a) of the Bundle Lemma, then the set **L** of all subalgebras of $L = \bigcup A$, $A \in \mathbf{B}$, satisfies conditions (i)–(v) of the definition of a logical structure. If, moreover, condition (b) of the Bundle Lemma holds, then **L** is a logical structure and L is the logic associated with **L**.

Proof. Let us assume first that for a family **B** of Boolean algebras version (a) of the Bundle Lemma holds. Let us form $L = \bigcup A$, $A \in \mathbf{B}$, with partial order and orthocomplementation defined as in the Bundle Lemma, and let us denote by **L** the set of all Boolean subalgebras of L. Conditions (i) and (iv) of the definition of a logical structure are obviously satisfied. Since the partial order relation in all subalgebras of L is inherited from L, (ii) is satisfied as well. For the same reason, instead of $x \leq_\alpha y$ and $y \leq_\beta z$, we can simply write $x \leq y$ and $y \leq z$, where $\leq$ is the partial order in L, so by the transitivity of $\leq$, $x \leq z$. Since the partial order $\leq$ in L is defined as the union of all relations $\leq_A$, $A \in \mathbf{B}$, we infer that there is a Boolean algebra A in **B**, and therefore also in **L**, such that $x, z \in A$ and $x \leq_A z$, i.e., (iii) holds. If $x, y \in A_\alpha \cap A_\beta$, then, since both A_α and A_β are Boolean subalgebras of L, there exist joins $x \vee_\alpha y$ and $x \vee_\beta y$ and they both coincide with a join $x \vee y$ of x and y in L, i.e., (v) holds.

Now, let us assume additionally that the condition (b) of the Bundle Lemma holds, so L is an orthomodular lattice. In such a case any pair of comparable elements $a \leq b$ or $a \geq b$ of L, as well as all pairs (a, b'), (a', b), and (a', b'), commute (see, e.g., Beltrametti and Cassinelli, 1981) and since partial order in all A_α is inherited from L, all pairs (x, y), (x, z), and (y, z) encountered in (vi) commute. Thus, the triple (x, y, z) generates a Boolean subalgebra A_δ of L^4 and since obviously $A_\delta \in \mathbf{L}$ the proof of (vi) and of Theorem 1 is completed. ■

In all the ways mentioned so far of joining together Boolean algebras (horizontal sums, pasting, utilization of the Loop Lemma or Bundle Lemma, logics associated with logical structures), any pair (a, b) of comparable elements of the emerging structure $L = \bigcup A$ existed already as a comparable pair in at least one of the Boolean algebras which formed L. In our constructive approach to quantum logics this means that propositions a and b were tested simultaneously in at least one single experiment. Therefore, only simultaneously testable pairs of propositions could be comparable, which generally is not the case in axiomatic quantum logics. However, let us note that if $a, b \in A$, $a \leq_A b$, and $c, d \in B$, $c \leq_B d$, and if we join A with B by identifying elements b and c, then we have $a \leq_A b = c \leq_B d$, so it is very natural to expect that $a \leq d$ in the final structure L, despite the fact that a and d do not belong to the same Boolean algebra, i.e., were not tested in a single experiment. The described procedure is known as the *transitive closure* (see, e.g., Kalmbach, 1983) of the union of relations $\leq_A$, $A \in \mathbf{B}$, and it will be used in the next section to form one set of propositions from four Boolean algebras obtained in four EPR experiments.

Let us note that the forming of the transitive closure of the union of experimentally established relations can be used for testing if identified elements from different Boolean algebras were chosen in a proper way. For example, if $a \leq_A b$ and $c \leq_B d$ and we are sure that $a \neq b$ and $c \neq d$, we cannot identify a with d and b with c, since this would yield $a = b = c = d$. Similarly, if $a \neq \emptyset$ and $a \leq_A b$, $c \leq_B d$, $e \leq_A a'$, the identification of b with c, and d with e, would yield $a \leq a'$, which, for $a \neq \emptyset$ should be impossible. Also, in the case $a \neq b$ and $c \leq_A a \leq_A d$, $e \leq_A b \leq_A f$, $g \leq_B h$, $k \leq_B m$ we should not identify c with h, g with f, d with k, and e with m, since this would imply $a = b$.

The above-mentioned examples show that the problem of joining together Boolean algebras by identifying elements and forming the transitive closure of the union of order relations is, when formulated in general terms, subtle and worth further study. Fortunately, in the constructive approach which we advocate in this paper, one always deals with finite families of finite Boolean algebras, so the process of joining them together and checking properties of the emerging structure can be computerized. In fact, there already exist computer programs designed for this purpose. For example, the user of the program described in Chapter 5(20) of Kalmbach (1983) can decide which elements of the input Boolean algebras should be identified and then the program checks if assumptions of the Bundle Lemma are fulfilled. Actually, in testing the transitivity of the partial order relation, its transitive closure is made, but a warning is printed out to the user indicating the elements for which there was no transitivity in the family of original relations. When the transitive closure is formed (or there is no need to do it), the program finds the join of every pair of elements or informs the user of the pairs for which they do not exist. If joins of all pairs of elements are found, the emerging structure, by the Bundle Lemma, is an orthomodular lattice.

Before closing this section, let us mention that some families of joined together Boolean algebras were regarded as appropriate models for physical theories also by Kochen and Specker (1965, 1967), who called them *partial Boolean algebras*, and also by Lock and Hardegree (1984). A *Boolean manifold* was defined by Lock and Hardegree as a family $(B_i, i \in I)$ of Boolean algebras which satisfies the following properties:

(i) If B_i is contained in B_j, then $B_i = B_j$.
(ii) If $a, b \in B_i \cap B_j$, then $a \leq_i b$ if and only if $a \leq_j b$.
(iii) $1_i = 1_j$, $\emptyset_i = \emptyset_j$ for all $i, j \in I$.
(iv) If $a \in B_i \cap B_j$, then $a'^i = a'^j$.
(v) If $a, b \in B_i \cap B_j$, then $a \wedge_i b = a \wedge_j b$ and $a \vee_i b = a \vee_j b$.

Lock and Hardegree have shown that this notion generalizes both the notion of Kochen and Specker's partial Boolean algebra and of an orthomodular partially ordered set. The study of similarities and differences between Lock and Hardegree's Boolean manifolds and Finch's (1969) logical structures is left to the reader.

4. LOGICS OF PROPOSITIONS OF EPR-TYPE EXPERIMENTS

In this section we shall build in a constructive way partially ordered sets of *experimental* propositions of an EPR-type experiment (Einstein *et al.*, 1935). It should be mentioned that within the quantum logic approach similar structures of *theoretical* propositions were studied recently by Szabó (1988) and Bub (1989). We shall compare their results with ours at the end of this section.

By an EPR-type experiment we mean an experiment in which a source S emits particles always in pairs in such a way that, after some time, members of each pair are spatially separated. Let us assume that particle 1 goes to the left, particle 2 to the right, and that on each side we can perform experiments testing one of two propositions a or b. These propositions are assumed not to be simultaneously testable for each of the particles separately, i.e., we cannot test simultaneously a_1 and b_1 or a_2 and b_2, where the subscript indicates on which particle the test is performed. Nevertheless, it is assumed that we can test simultaneously the following four pairs of propositions: (a_1, a_2), (a_1, b_2), (b_1, a_2), and (b_1, b_2). Of course, this can be done only in four different experiments.

The identification of propositions in EPR-type experiments rests upon the assumption of locality, that is, the hypothesis that a modification introduced in some region of space cannot have any influence in another region spatially separated (in the sense of relativity theory) from the first one. Locality is noncontextuality for EPR-type experiments, i.e., the assumption that two measurements performed at spatially separate regions cannot influence each other.

4.1. EPR Experiment with Random Source and Ideal Counters

Let us assume now that testing of a and b is done with the aid of ideal counters which fire or not, allowing one to attach the answer "yes" = 1 or "no" = 0 to a and b, and that the source emits pairs of particles at random. As we shall see, the second assumption imposes severe restrictions on the structure of the set of experimentally verifiable propositions. In fact, such a situation is generally not studied in axiomatic approaches to quantum logics, where, in the case of a yes–no experiment, it is assumed that one collects data being sure that the experiment was actually performed, so both a positive and a negative result of any test yields information about a physical system. Contrary to this, in the investigated situation we cannot experimentally distinguish the case of having negative results for both x_1 and x_2, $x = a$ or b (none of the counters fires), from the case that, during the time of observation, a pair of particles was not emitted. This second possibility means the nonexistence of a copy of our physical system in this particular run of the experiment; therefore, all cases when none of the counters fires are identified with obtaining a negative result in the test of a trivial proposition I = "the system exists." Therefore, we are bound to count only those runs of experiments in which at least one of the counters testing x_1 or x_2 fires, since only they convey information that the system actually exists. The sequences of 0's and 1's representing nontrivial propositions in any of the four experiments testing pairs (x_1, x_2) might be, for instance, as in Table I.

Note the absence of pairs (0, 0) in the first two rows of Table I. Note also that, according to formula (3), sequences representing propositions x_1' and x_2' are obtained from sequences representing x_1 and x_2 by replacing all 0's by 1's and vice versa. In fact, x_1' and x_2' are tested by the same counters which test x_1 and x_2. As we shall see in Example 2, if x_1' and x_2' can be checked by independent counters, quite a different family of Boolean algebras emerges.

There are only three different columns in Table I, for example, the first three. Actually, these columns, or even their entries for x_1 and x_2, define the whole Boolean algebra uniquely and this algebra is isomorphic to the Boolean algebra $\mathbf{2}^3$ (see Figure 1).

The whole structure obtained by identification, respectively, of propositions a_i and b_i, $i = 1, 2$, as well as their orthocomplements, appearing in different Boolean algebras $\mathbf{2}^3$ is presented in Figure 2.

It can be checked that all assumptions of the Loop Lemma are fulfilled and that the structure of Figure 2 consists in fact of a single loop of the order 4. Thus, it is an orthomodular poset, but it is not an orthomodular

Table I. Possible Results of EPR-Type Experiments with Random Source

	Result for runs 1, 2, ...												
Proposition	1	2	3	4	5	6	7	8	9	10	11	12	...
x_1	1	0	1	0	1	0	1	0	1	1	1	0	...
x_2	0	1	1	1	1	1	1	1	0	0	1	1	...
x_1'	0	1	0	1	0	1	0	1	0	0	0	1	...
x_2'	1	0	0	0	0	0	0	0	1	1	0	0	...
$x_1 \wedge x_2$	0	0	1	0	1	0	1	0	0	0	1	0	...
$x_1' \vee x_2'$	1	1	0	1	0	1	0	1	1	1	0	1	...

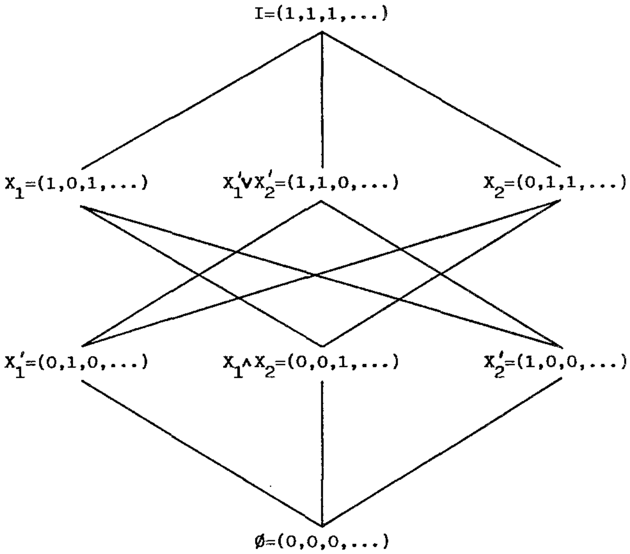
I=(1,1,1,...)
X1=(1,0,1,...)
X1'vX2'=(1,1,0,...)
X2=(0,1,1,...)
X1'=(0,1,0,...)
X1∧X2=(0,0,1,...)
X2'=(1,0,0,...)
∅=(0,0,0,...)

Fig. 1

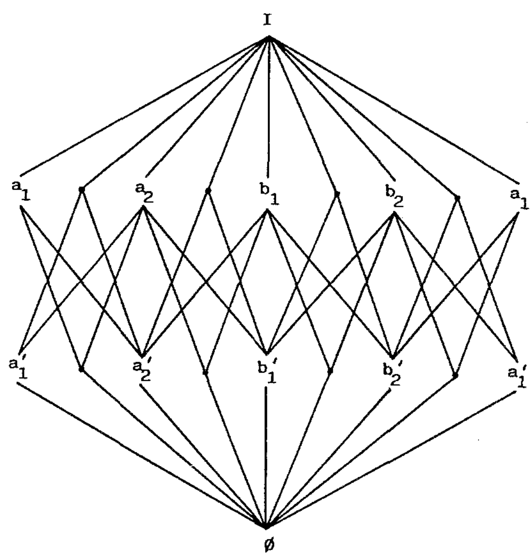
I
a1
a2
b1
b2
a1
a1'
a2'
b1'
b2'
a1'
∅

Fig. 2

lattice. The conclusion that the structure of Figure 2 is not a lattice also can be obtained as well by noticing that $a_2' \le a_1, b_1$ and $b_2' \le a_1, b_1$, but since a_1 and b_1 are not comparable, none of them is the join of a_2' and b_2'.

4.2. EPR Experiment with Controllable Source

Let us assume now that source S which emits pairs of particles is controllable in the sense that we can actually state in the course of every single observation whether a pair of particles was emitted or not. This does not have to be done necessarily by forcing the source to emit pairs of particles only when we want it. It can be achieved, for example, by testing actually not only propositions a_i or b_i, but also their complements a_i' or b_i', since in such a case only the response "no" for both a proposition and its complement would imply that a particle was not emitted. Theoretically, the counter testing x_i', $x = a, b$, $i = 1, 2$, should detect the presence of a particle in the whole space available to the particle and not occupied by the counter which measures x_i. Of course, again counters have to be "ideal," as well as all the rest of the experimental device, which should be "ideal," too. The alternative way of testing whether the act of emission actually took place (and at the same time of testing actually x_i') could be done by a negative-result measurement as described by Namiki (1986). In this case we should place another counter D_0 (a nondestructive and ideal one) just after the source, so the anticoincidence of firing of D_0 and of the counter which tests x_i would yield the required information about x_i'.

The sequences of 0's and 1's representing nontrivial propositions in any of the four experiments testing pairs (x_1, x_2) might be, in the present case, as in Table II.

Therefore, the Boolean algebra of experimental propositions obtained for each experiment is now isomorphic to the Boolean algebra $\mathbf{2}^4$, which is drawn in Figure 3.

To obtain all four specific Boolean algebras encountered in this version of the EPR experiment, it suffices to replace x_1 and x_2, respectively, by pairs (a_1, a_2), (a_1, b_2), (b_1, a_2), and (b_1, b_2). We shall denote these four algebras $A(x_1, x_2)$, $x = a, b$. Now let us form from these four algebras the united structure L by identifying elements bearing the same label. The structure L has $4 \times 16 - 4 \times 2 - 2 \times 3 = 50$ elements [we subtract four pairs of the form (x_1, x_2), three $\varnothing$'s, and three $\imath$'s to avoid counting them more than once]. It can be easily seen that the bare union of all already existing partial order relations is not transitive on L. For example,

$$a_1 \wedge a_2 \le_{a_1 a_2} a_1 \le_{a_1 b_2} a_1 \vee b_2$$

but $a_1 \wedge a_2$ and $a_1 \vee b_2$ are not comparable yet. Therefore, it is necessary to

Table II. Possible Results of EPR-Type Experiments with Controllable Source

Proposition	Result for runs 1, 2, ...															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
x_1	1	1	0	0	1	0	0	1	1	0	1	0	1	0	0	...
x_2	1	0	1	0	0	1	0	0	1	1	0	1	0	0	1	...
x_1'	0	0	1	1	0	1	1	0	0	1	0	1	0	1	1	...
x_2'	0	1	0	1	1	0	1	1	0	0	1	0	1	1	0	...
$x_1 \wedge x_2$	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	...
$x_1 \wedge x_2'$	0	1	0	0	1	0	0	1	0	0	1	0	1	0	0	...
$x_1' \wedge x_2$	0	0	1	0	0	1	0	0	0	1	0	1	0	0	1	...
$x_1' \wedge x_2'$	0	0	0	1	0	0	1	0	0	0	0	0	0	1	0	...
$x_1 \vee x_2$	1	1	1	0	1	1	0	1	1	1	1	1	1	0	1	...
$x_1 \vee x_2'$	1	1	0	1	1	0	1	1	1	0	1	0	1	1	0	...
$x_1' \vee x_2$	1	0	1	1	0	1	1	0	1	1	0	1	0	1	1	...
$x_1' \vee x_2'$	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	...
$(x_1 \vee x_2) \wedge (x_1' \vee x_2')$	0	1	1	0	1	1	0	1	0	1	1	1	1	0	1	...
$(x_1 \wedge x_2) \vee (x_1' \wedge x_2')$	1	0	0	1	0	0	1	0	1	0	0	0	0	1	0	...

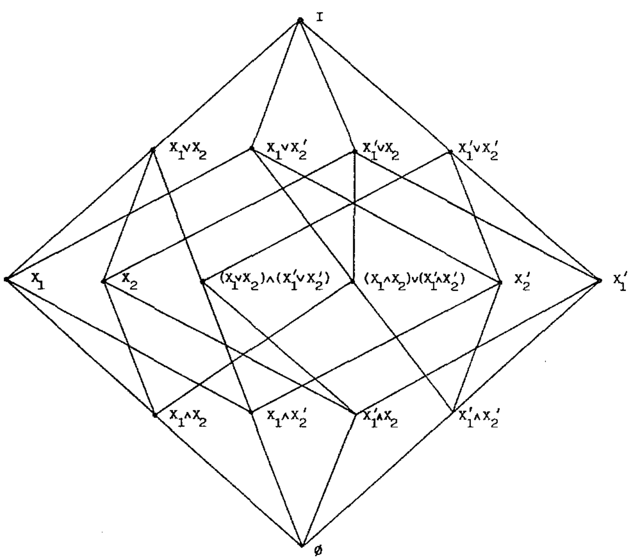

Fig. 3

form the transitive closure of the union of existing relations. It can be checked that no difficulties of the type mentioned at the end of the previous section are met in the present construction. In fact, overlapping of neighboring algebras is now as minimal as in the previous example: two neighboring algebras have, besides $\varnothing$ and I, only one element and its orthocomplement in common. However, now this element is not an atom and the total number of atoms in passing from the family of Boolean algebras **B** to the unified structure L is not diminished.

Finally, let us mention that, as in the previous example, the obtained structure L is not a lattice. For example,

$$b_1 \wedge a_2 \leq b_1 \leq b_1 \vee b_2' \geq b_2' \geq a_1 \wedge b_2' \leq a_1 \leq a_1 \vee a_2 \geq a_2 \geq b_1 \wedge a_2$$

so both $b_1 \wedge a_2$ and $a_1 \wedge b_2'$ are dominated by $a_1 \vee a_2$ and $b_1 \vee b_2'$ [$\wedge$ and $\vee$ are meets and joins in Boolean algebras $A(x_i, x_j)$, not in a partially ordered set L]. However, they are not comparable and since in fact there are no other elements, besides $\varnothing$, dominated simultaneously by $a_1 \vee a_2$ and $b_1 \vee b_2'$, neither $b_1 \wedge a_2$ nor $a_1 \wedge b_2'$ is the meet of $a_1 \vee a_2$ and $b_1 \vee b_2'$ in the whole partially ordered set L. Moreover, L is not an orthomodular and even not orthocomplete partially ordered set, since, for example, $b_1 \wedge a_2' \leq b_1 \vee b_2 = (b_1' \wedge b_2')'$, so $b_1 \wedge a_2' \perp b_1' \wedge b_2'$, but their join does not exist in L.

The fact that L is not a lattice does not mean that L cannot be recognized as a substructure of the theoretical lattice of closed subspaces or, equivalently, projectors on such subspaces, of the tensor product of two two-dimensional Hilbert spaces of the EPR-type experiment. However, the sublattice of projectors generated by 16 atoms of the form

$$P_{x_1}P_{x_2}, \quad P_{x_1}(1-P_{x_2}), \quad (1-P_{x_1})P_{x_2}, \quad (1-P_{x_1})(1-P_{x_2}), \qquad x = a, b \tag{11}$$

has more elements than L. It can be checked, for example, that elements of such a sublattice which correspond to the elements $a_1 \vee a_2$ and $b_1 \vee b_2'$ mentioned in the previous counterexample are, respectively, projectors $P_{a_1} + P_{a_2} - P_{a_1}P_{a_2}$ and $1 - P_{b_2} + P_{b_1}P_{b_2}$. Their meet in the sublattice of projectors has to exist. However, it does not correspond to any element in our experimental poset L.

The 16-element Boolean algebra, consisting of projectors onto closed subspaces of the tensor product of two two-dimensional Hilbert spaces of the EPR experiment and generated by four atoms of the form (11) for a fixed pair (x_1, x_2), was obtained by Bub (1989) in a purely theoretical way. It is of course isomorphic to our Boolean algebra $\mathbf{2}^4$ consisting of propositions of the "one-fourth" of the EPR experiment (Figure 3), since all finite Boolean algebras with the same number of elements are isomorphic (see, e.g., Halmos, 1974). Bub has not investigated the whole sublattice generated by 16 projectors of the form (11). According to our calculations such a sublattice should consist of 146 elements: $\varnothing$ (projector onto empty subspace), ɪ (projector onto the whole space), 16 atoms of the form (11), 16 coatoms which are orthocomplements of atoms (11), and 112 projectors onto two-dimensional subspaces which are joins of pairs of atoms (equivalently: meets of pairs of coatoms) yielding different elements. To make the comparison of number of elements complete, recall that the structure L of experimentally obtainable propositions of an EPR-type experiment with controllable source has 50 elements, while the Boolean algebra generated by 16 different elements has $2^{16} = 65{,}536$ elements placed on 17 different levels in comparison to five levels of the structure L of EPR sublattice of the lattice of projectors (projectors onto $0, 1, 2, 3$, and four-dimensional subspaces).

As far as Szabó's (1988) results are concerned, we think that his 20-element but five-level structure presented on Figure 1 of his paper, in view of our Examples 1 and 2, is too small to describe the whole set of (theoretical) EPR propositions encountered in the experiment with the controllable source and, on the other hand, it has too many levels to describe the set of propositions of the EPR experiment with random source. Moreover, although he calls the structure represented on his Figure 1 "the smallest non-Boolean model quantum lattice describing the EPR events," it can be easily checked that this nonsymmetric structure is only a partially ordered set, not a lattice. [Note the difference of this structure and another 20-element symmetric structure which is presented on Figure 2 of the other paper of Szabó (1987). The latter *is* a non-Boolean orthocomplemented lattice; however, Szabó does not claim that it describes EPR events.] In fact, Szabó's (1988) structure, apart from the four elements (denoted on his drawing by letters E, g, h, and m), is our familiar Boolean algebra $\mathbf{2}^4$, which describes "one-fourth" of the full EPR-type experiment with controllable source.

Let us finish with the reflection that the situation studied in Example 1 seems to be closer to real EPR-type experiments than that of Example 2, which in turn is more similar to theoretical structures obtained within the Hilbert space quantum formalism by Bub (1989). However, since the partially ordered set of Example 2 is not even orthocomplete, the usual quantum logic notion of a state defined to be a probability measure on the logic of propositions cannot be applied in this case. We suspect that the more general "empirical logic" approach of Foulis and Randall (see Foulis and Randall, 1972; Randall and Foulis, 1973; Kläy, 1988) in which states are not represented by probability measures on partially ordered orthocomplete sets would be more suitable here. An empirical logic approach was already applied by Kläy (1988) to theoretical studies of the EPR case and its application to the EPR case within the framework of the constructive approach will be the objective of a forthcoming paper.

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